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LETTER TO THE EDITOR

Non-relativistic quantum mechanics in a non-commutative space

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Abstract. Non-relativistic quantum theory of a particle in a non-commutative space is formulated within the framework of the usual quantum mechanics itself. This is accomplished by showing explicitly that the Wess–Zumino formalism of deformed quantum mechanical phase space corresponding to a non-commutative space can be realized, in principle, in terms of the usual quantum mechanical phase space. The case of a slightly non-commutative space is considered in detail. A free particle in such a space is found to behave as if it is a charged particle moving in a self-generated magnetic field.

There is a growing interest in the study of non-commutative differential calculus with a view to exploring the possibilities of its role in physical phenomena. In this letter we analyse the effect of non-commutativity of the basic coordinate space on the non-relativistic dynamics of a quantum particle. We use the Wess–Zumino formalism of non-commutative differential calculus [1] (see also [2]) and certain approximations of an explicit realization of it. Essentially, we show that it is possible to take into account the non-commutativity of the underlying coordinate space, if it exists at all, within the framework of the usual quantum mechanics itself. To this end, we make use of the fact that the non-commutative coordinates can be expressed in terms of a set of an equal number of commuting coordinates through an invertible map [3, 4]. The recent discovery [4] of the explicit form of the inverse map giving the commuting coordinates completely in terms of the non-commutative coordinates, guarantees, in principle, the validity of our theory.

Following Wess and Zumino [1] let us consider the non-commutative space (NCS) with *real coordinates* (X, Y, Z) *satisfying the commutation relations*

$$XY = qYX \qquad XZ = qZX \qquad YZ = qZY \qquad q = e^{i\theta}. \tag{1}$$

The reality of (X, Y, Z) implies that they should be realizable as Hermitian operators. The NCS coordinates (X, Y, Z) and the corresponding partial derivatives

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$(\partial_X, \partial_Y, \partial_Z)$ obey the non-commutative calculus [1]

$$
\begin{aligned}
&XY = qYX \qquad && XZ = qZX \qquad && YZ = qZY \\
&\partial_X Y = qY\partial_X && \partial_X Z = qZ\partial_X && \\
&\partial_Y X = qX\partial_Y && \partial_Y Z = qZ\partial_Y && \\
&\partial_Z X = qX\partial_Z && \partial_Z Y = qY\partial_Z && \\
&\partial_X\partial_Y = \bar{q}\partial_Y\partial_X && \partial_X\partial_Z = \bar{q}\partial_Z\partial_X && \partial_Y\partial_Z = \bar{q}\partial_Z\partial_Y \\
&\partial_X X - q^2 X\partial_X = 1 + (q^2-1)(Y\partial_Y + Z\partial_Z) && && \\
&\partial_Y Y - q^2 Y\partial_Y = 1 + (q^2-1)Z\partial_Z && && \\
&\partial_Z Z - q^2 Z\partial_Z = 1. && &&
\end{aligned}
\tag{2}
$$

Writing $X = X_1$, $Y = X_2$, $Z = X_3$, $\partial_X = \partial_1$, $\partial_Y = \partial_2$, $\partial_Z = \partial_3$, the above relations (2) can be stated as

$$
\begin{aligned}
&X_iX_j = qX_jX_i \qquad \partial_i\partial_j = \bar{q}\partial_j\partial_i \qquad i<j \\
&\partial_i X_j = qX_j\partial_i \qquad i \neq j \\
&\partial_i X_i - q^2 X_i\partial_i = 1 + (q^2-1)\sum_{j>i} X_j\partial_j \qquad i = 1,2 \\
&\partial_3 X_3 - q^2 X_3\partial_3 = 1.
\end{aligned}
\tag{3}
$$

An explicit realization of the non-commutative (or 'quantum') space coordinates (X, Y, Z) in terms of the commutative (or 'classical') space coordinates (x, y, z) is as follows [3, 4]

$$
\begin{aligned}
X &= x\left\{\frac{q^{2(N_x+1)}-1}{(q^2-1)(N_x+1)}\right\}^{1/2} q^{(N_y+N_z)} \\
Y &= y\left\{\frac{q^{2(N_y+1)}-1}{(q^2-1)(N_y+1)}\right\}^{1/2} q^{N_z} \\
Z &= z\left\{\frac{q^{2(N_z+1)}-1}{(q^2-1)(N_z+1)}\right\}^{1/2} \\
N_x &= x\partial_x \qquad N_y = y\partial_y \qquad N_z = z\partial_z
\end{aligned}
\tag{4}
$$

where $(\partial_x, \partial_y, \partial_z)$ are the partial derivatives with respect to (x, y, z) respectively. The corresponding realization of the derivatives $(\partial_X, \partial_Y, \partial_Z)$ is given [3, 4] by the formulae

$$\partial_X = q^{(N_y+N_z)}\left\{\frac{q^{2(N_x+1)}-1}{(q^2-1)(N_x+1)}\right\}^{1/2}\partial_x$$
$$\partial_Y = q^{N_z}\left\{\frac{q^{2(N_y+1)}-1}{(q^2-1)(N_y+1)}\right\}^{1/2}\partial_y \tag{5}$$
$$\partial_Z = \left\{\frac{q^{2(N_z+1)}-1}{(q^2-1)(N_z+1)}\right\}^{1/2}\partial_z .$$

This realization (4), (5) is seen to satisfy the relations (2) purely by virtue of the algebraic relations between $(x, y, z, \partial_x, \partial_y, \partial_z, N_x, N_y, N_z)$. Note that there can be other equivalent realizations [4, 5], but we have chosen the one above.

Based on an earlier result [6] (see also [7]) in the one-dimensional case, the inverse map $(X, Y, Z) \to (x, y, z)$ can be written [4] completely in terms of (X, Y, Z)

$$x = X\bar{q}^{(N_Y+N_Z)}\left\{\frac{q^{2(N_X+1)}-1}{(q^2-1)(N_X+1)}\right\}^{-1/2}$$
$$y = Y\bar{q}^{N_Z}\left\{\frac{q^{2(N_Y+1)}-1}{(q^2-1)(N_Y+1)}\right\}^{-1/2} \tag{6}$$
$$z = Z\left\{\frac{q^{2(N_Z+1)}-1}{(q^2-1)(N_Z+1)}\right\}^{-1/2}$$

and

$$\partial_x = \bar{q}^{(N_Y+N_Z)}\left\{\frac{q^{2(N_X+1)}-1}{(q^2-1)(N_X+1)}\right\}^{-1/2}\partial_X$$
$$\partial_y = \bar{q}^{N_Z}\left\{\frac{q^{2(N_Y+1)}-1}{(q^2-1)(N_Y+1)}\right\}^{-1/2}\partial_Y \tag{7}$$
$$\partial_z = \left\{\frac{q^{2(N_Z+1)}-1}{(q^2-1)(N_Z+1)}\right\}^{-1/2}\partial_Z$$

where

$$N_X = \sum_{k=1}^{\infty}\frac{(1-q^2)^k}{(1-q^{2k})}X^k(\partial_X)^k\bar{q}^{2k(N_Y+N_Z)}$$
$$N_Y = \sum_{k=1}^{\infty}\frac{(1-q^2)^k}{(1-q^{2k})}Y^k(\partial_Y)^k\bar{q}^{2kN_Z} \tag{8}$$
$$N_Z = \sum_{k=1}^{\infty}\frac{(1-q^2)^k}{(1-q^{2k})}Z^k(\partial_Z)^k .$$

It may be noted that (for details see [4])

$$
\begin{aligned}
&[N_X, X] = X \qquad [N_Y, Y] = Y \qquad [N_Z, Z] = Z \\
&[N_X, \partial_X] = -\partial_X \qquad [N_Y, \partial_Y] = -\partial_Y \qquad [N_Z, \partial_Z] = -\partial_Z \\
&[N_X, Y] = [N_X, Z] = [N_Y, X] = [N_Y, Z] = [N_Z, X] = [N_Z, Y] = 0 \\
&[N_X, \partial_Y] = [N_X, \partial_Z] = [N_Y, \partial_X] = [N_Y, \partial_Z] = [N_Z, \partial_X] = [N_Z, \partial_Y] = 0 \\
&[N_X, N_Y] = [N_X, N_Z] = [N_Y, N_Z] = 0.
\end{aligned}
\tag{9}
$$

It is seen that (x, y, z) and $(\partial_x, \partial_y, \partial_z)$ as realized by (6) and (7), respectively, obey the usual differential calculus by virtue of the algebraic relations (2) and (9). The inverse maps (6) and (7) are also given in [3]; but, only formally, without the explicit realization of (N_X, N_Y, N_Z) as in (8).

Let us now approximate the above realization of (X, Y, Z) and $(\partial_X, \partial_Y, \partial_Z)$ assuming θ to be almost zero. Among the terms differing only in their c-number coefficients we shall retain only those proportional to the lowest power of θ. Then, up to first order in θ, we can write

$$
\begin{aligned}
X &\approx x + \mathrm{i}\theta\left(x\left[\{y\partial_y\}_s + \{z\partial_z\}_s\right] + \tfrac{1}{2}\{x^2\partial_x\}_s\right) \\
Y &\approx y + \mathrm{i}\theta\left(y\{z\partial_z\}_s + \tfrac{1}{2}\{y^2\partial_y\}_s\right) \\
Z &\approx z + \tfrac{1}{2}\mathrm{i}\theta\{z^2\partial_z\}_s
\end{aligned}
\tag{10}
$$

and

$$
\begin{aligned}
\partial_X &\approx \partial_x + \mathrm{i}\theta\left(\partial_x\left[\{y\partial_y\}_s + \{z\partial_z\}_s\right] + \tfrac{1}{2}\{x(\partial_x)^2\}_s\right) \\
\partial_Y &\approx \partial_y + \mathrm{i}\theta\left(\partial_y\{z\partial_z\}_s + \tfrac{1}{2}\{y(\partial_y)^2\}_s\right) \\
\partial_Z &\approx \partial_z + \tfrac{1}{2}\mathrm{i}\theta\{z(\partial_z)^2\}_s
\end{aligned}
\tag{11}
$$

where $\{\ldots\}_s$ denotes symmetrized expressions in (x, ∂_x), (y, ∂_y) and (z, ∂_z) such that (X, Y, Z) and $(-\mathrm{i}\hbar\partial_X, -\mathrm{i}\hbar\partial_Y, -\mathrm{i}\hbar\partial_Z)$ are Hermitian operators.

Let us now consider that for the quantum mechanics of a particle in a NCS with an infinitesimally small value for θ, the coordinate operators and the corresponding momentum operators are respectively (X, Y, Z) and $(P_X = -\mathrm{i}\hbar\partial_X, P_Y = -\mathrm{i}\hbar\partial_Y, P_Z = -\mathrm{i}\hbar\partial_Z)$ as given by (10) and (11). Then, we can write

$$
\begin{aligned}
X &\approx x - \frac{\theta}{\hbar}\left(x\left[\{yp_y\}_s + \{zp_z\}_s\right] + \tfrac{1}{2}\{x^2p_x\}_s\right) \\
Y &\approx y - \frac{\theta}{\hbar}\left(y\{zp_z\}_s + \tfrac{1}{2}\{y^2p_y\}_s\right) \\
Z &\approx z - \frac{\theta}{2\hbar}\{z^2p_z\}_s
\end{aligned}
\tag{12}
$$

and

$$
\begin{aligned}
P_X &\approx p_x - \frac{\theta}{\hbar}\left(p_x\left[\{yp_y\}_s + \{zp_z\}_s\right] + \tfrac{1}{2}\{xp_x^2\}_s\right) \\
P_Y &\approx p_y - \frac{\theta}{\hbar}\left(p_y\{zp_z\}_s + \tfrac{1}{2}\{yp_y^2\}_s\right) \\
P_Z &\approx p_z - \frac{\theta}{2\hbar}\{zp_z^2\}_s
\end{aligned}
\tag{13}
$$

where $(p_x = -i\hbar\partial_x, p_y = -i\hbar\partial_y, p_z = -i\hbar\partial_z)$ are the usual momentum operators conjugate to the commuting coordinates (x, y, z). This is essentially the same as the suggestion of Wess and Zumino [1]; here, due to the approximations made and the process of symmetrization adopted to make the NCS coordinate and momentum operators Hermitian the q-factors appearing in the Wess–Zumino formulae are not seen explicitly. Actually, the q-factors appearing in the Wess–Zumino formulae correspond, in a similar way, also to a choice of realization for (X, Y, Z) and $(\partial_X, \partial_Y, \partial_Z)$ involving the Jackson q-derivatives [4, 5] instead of the above realization (4), (5).

Now, the Hamiltonian operator of a quantum particle moving in such a slightly non-commutative space (SNCS) may be considered to be derived from the corresponding classical Hamiltonian by the usual replacement rule

$$
H(x, y, z, p_x, p_y, p_z) \longrightarrow \mathcal{H}(X, Y, Z, P_X, P_Y, P_Z). \tag{14}
$$

The SNCS coordinate operators (X, Y, Z) and momentum operators (P_X, P_Y, P_Z) are seen to obey the following deformed Heisenberg commutation relations: up to first order in θ

$$
\begin{aligned}
&[X, Y] \approx i\theta YX \qquad [X, Z] \approx i\theta ZX \qquad [Y, Z] \approx i\theta ZY \\
&[P_X, Y] \approx i\theta YP_X \qquad [P_X, Z] \approx i\theta ZP_X \\
&[P_Y, X] \approx i\theta XP_Y \qquad [P_Y, Z] \approx i\theta ZP_Y \\
&[P_Z, X] \approx i\theta XP_Z \qquad [P_Z, Y] \approx i\theta YP_Z \\
&[P_X, P_Y] \approx -i\theta P_Y P_X \qquad [P_X, P_Z] \approx -i\theta P_Z P_X \qquad [P_Y, P_Z] \approx -i\theta P_Z P_Y \\
&[X, P_X] \approx i\hbar - 2i\theta\,(XP_X + YP_Y + ZP_Z) \\
&[Y, P_Y] \approx i\hbar - 2i\theta\,(YP_Y + ZP_Z) \\
&[Z, P_Z] \approx i\hbar - 2i\theta\, ZP_Z .
\end{aligned}
\tag{15}
$$

The expressions in (12) and (13) provide Hermitian realizations of (X, Y, Z) and (P_X, P_Y, P_Z), satisfying the relations (15), in terms of the usual quantum mechanical operators (x, y, z, p_x, p_y, p_z). The inverse relationships (6) and (7) reduce to the following approximate realizations of (x, y, z) and (p_x, p_y, p_z) in terms of (X, Y, Z, P_X, P_Y, P_Z)

$$
\begin{aligned}
x &\approx X + \frac{\theta}{\hbar}(X[\{YP_Y\}_s + \{ZP_Z\}_s] + \tfrac{1}{2}\{X^2P_X\}_s) \\
y &\approx Y + \frac{\theta}{\hbar}(Y\{ZP_Z\}_s + \tfrac{1}{2}\{Y^2P_Y\}_s) \\
z &\approx Z + \frac{\theta}{2\hbar}\{Z^2P_Z\}_s
\end{aligned}
\tag{16}
$$

and

$$
\begin{aligned}
p_x &\approx P_X + \frac{\theta}{\hbar}(P_X[\{YP_Y\}_s + \{ZP_Z\}_s] + \tfrac{1}{2}\{XP_X^2\}_s) \\
p_y &\approx P_Y + \frac{\theta}{\hbar}(P_Y\{ZP_Z\}_s + \tfrac{1}{2}\{YP_Y^2\}_s) \\
p_z &\approx P_Z + \frac{\theta}{2\hbar}\{ZP_Z^2\}_s.
\end{aligned}
\tag{17}
$$

It can be easily verified that these expressions of (x, y, z) and (p_x, p_y, p_z) in terms of (X, Y, Z) and (P_X, P_Y, P_Z) are consistent with the usual Heisenberg commutation relations up to first order in θ. If the quantum mechanical Ψ-function of the system is given in terms of the NCS coordinates it can involve only one of (X, Y, Z) or some single function of them. However, since (x, y, z) defined by (6) (or (16), approximately, up to first order in θ) form a complete set of commuting operators the state of the system can be represented by a $\Psi(x, y, z)$. This means that we can continue to describe the system in terms of the usual quantum mechanics with reference to a frame of regular Cartesian coordinates (x, y, z) which may be identified with the commuting 'position observables'. The effect of non-commutativity of the underlying (X, Y, Z)-coordinate space is manifested in the deformation of the Hamiltonian $H \longrightarrow \mathcal{H}$. The Schrödinger equation for the system would be as usual

$$
i\hbar\frac{\partial\Psi(x, y, z, t)}{\partial t} = \mathcal{H}\Psi(x, y, z, t). \tag{18}
$$

With the above understanding, the non-relativistic Schrödinger equation for a free particle of mass m moving in a SNCS can be written as

$$
i\hbar\frac{\partial\Psi}{\partial t} = \frac{1}{2m}(P_X^2 + P_Y^2 + P_Z^2)\Psi \tag{19}
$$

where (P_X, P_Y, P_Z) are given by (13). Here, we shall not be interested in an exact mathematical description of the dynamics of the system represented by (19); let us try to get an approximate, intuitive, physical picture. Since the non-commutativity of the space has been assumed to be infinitesimally small it is reasonable to expect a solution of (19) to be of the form

$$
\Psi(\boldsymbol{r}, t) = \phi_{\boldsymbol{k}}(\boldsymbol{r}) \exp\{i[\boldsymbol{k}\cdot\boldsymbol{r} - \omega_{\boldsymbol{k}}t]\} \tag{20}
$$

where $\phi_{\boldsymbol{k}}(\boldsymbol{r})$ should be a very slowly varying function of $\boldsymbol{r}$ such that

$$\left|\frac{-\mathrm{i}}{\phi_{\boldsymbol{k}}}\frac{\partial\phi_{\boldsymbol{k}}}{\partial x}\right| \ll |k_x| \qquad \left|\frac{-\mathrm{i}}{\phi_{\boldsymbol{k}}}\frac{\partial\phi_{\boldsymbol{k}}}{\partial y}\right| \ll |k_y| \qquad \left|\frac{-\mathrm{i}}{\phi_{\boldsymbol{k}}}\frac{\partial\phi_{\boldsymbol{k}}}{\partial z}\right| \ll |k_z|. \tag{21}$$

Then, it is straightforward to see that (19) can be approximated as

$$\begin{aligned}
&\mathrm{i}\hbar\frac{\partial\Psi}{\partial t} \approx \frac{1}{2m}\left(\mathcal{P}_X^2 + \mathcal{P}_Y^2 + \mathcal{P}_Z^2\right)\Psi \\
&\mathcal{P}_X = p_x - \theta\hbar k_x\left(\tfrac{1}{2}k_x x + k_y y + +k_z z\right) \\
&\mathcal{P}_Y = p_y - \theta\hbar k_y\left(\tfrac{1}{2}k_y y + k_z z\right) \\
&\mathcal{P}_Z = p_z - \tfrac{1}{2}\theta\hbar k_z^2 z.
\end{aligned} \tag{22}$$

The similarity between $(\mathcal{P}_X, \mathcal{P}_Y, \mathcal{P}_Z)$ in (22) and the kinetic momentum operators $\{p_x - (e/c)A_x,\ p_y - (e/c)A_y,\ p_z - (e/c)A_z\}$ of a particle of charge e moving in an external magnetic field is very striking. Using this analogy, we can say that the behaviour of a free particle in a SNCS, as described by the Schrödinger equation (19), would be like that of a particle with charge e moving in a self-generated magnetic field given by

$$B_x \approx -\frac{\hbar c}{e}\theta k_y k_z \qquad B_y \approx \frac{\hbar c}{e}\theta k_z k_x \qquad B_z \approx -\frac{\hbar c}{e}\theta k_x k_y. \tag{23}$$

It may be noted that if $\Psi(\boldsymbol{r}, t)$ depends on only one of the coordinates (x, y, z) then the particle does not feel any such magnetic field. Since the set of commutation relations (1) between the NCS coordinates does not possess the usual SO(3) covariance of the commutative space it is understandable that there exist preferred (x, y, z) axes corresponding to a given (X, Y, Z) frame.

In the case of an isotropic harmonic oscillator in a SNCS the Schrödinger equation (19) would correspond to an ordinary quantum mechanical oscillator with a perturbation which would remove the degeneracies in the energy spectrum. The isotropic harmonic oscillator in a NCS has been studied earlier [8] using the Wess–Zumino approach with (X, Y, Z) treated as abstract non-commuting objects and Ψ taken to be a function of an $\mathrm{SO}_q(3)$-invariant combination of (X, Y, Z). Such a study also leads to the discovery of the removal of degeneracies. In a similar treatment of the hydrogen atom [9] Ψ involves all the three coordinates (X, Y, Z) and is really an operator. In such treatments the meaning of Ψ, and the 'Schrödinger equation' obeyed by it, is not clear as has been pointed out in [8].

In conclusion, we can summarize our findings as follows. Using the idea of deformed quantum mechanical phase space corresponding to a NCS, proposed by Wess and Zumino [1], we have shown how to take into account the non-commutativity of the underlying coordinate space, effectively, within the framework of the usual quantum mechanics itself. The existence (equations (6)–(8) above) of an explicit realization [4] of a map [3, 4] from the NCS coordinates (X, Y, Z) to a complete set of commuting coordinates (x, y, z) guarantees that the necessary condition for the validity of our formalism, in principle, is satisfied. Treating the non-commutativity of the space to be infinitesimal we have formulated the non-relativistic quantum theory in such a space. It is found that the non-relativistic quantum mechanical behaviour of a free particle in such a SNCS would be as if it were an ordinary 'charged' quantum particle moving in a self-generated 'magnetic field'. In general, any non-relativistic quantum mechanical system in such a SNCS would behave like its counterpart in the commutative space in the presence of an additional system-dependent perturbation produced by the underlying NCS.

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